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Application of Laplace-Borel Transformation to the Representation of Analytical Solutions of Duffing's Equation

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Abstract: We discuss different features of the solutions of Duffing's equation using a representation of the solutions in the Laplace-Borel transform domain. Application of this technique is illustrated for the symmetry-breaking bifurcation of a hard spring.

APPLICATION OF THE LAPLACE-BOREL TRANSFORMATION TO THE REPRESENTATION OF ANALYTICAL SOLUTIONS OF DUFFING'S EQUATION

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SUMMARY

Various features of the solutions of Duffing's equation are described using a representation of the solutions in the Laplace-Borel transform domain. An application of this technique is illustrated for the symmetry-breaking bifurcation of a hard spring.

INTRODUCTION

For linear systems, it is well recognized that the operational calculus based on Laplace transformation is a convenient and powerful technique. One would like to have a similar operational calculus for treating nonlinear systems. Fliess and his coworkers (ref. 1) (cf. also ref. 2) advocate the use of functional expansions based on noncommutative power series. One practical consequence of the approach is the use of Laplace-Borel transformation in a similar way to that of ordinary Laplace transformation. In addition to the advantages of the Laplace transformation, the new transformation enables treatment of the product operation of functions via the so-called shuffle operation or mélange.

In this paper, we shall apply Laplace-Borel transformation to Duffing's equation. This simple system shows some rich features of nonlinear mechanics, e.g., existence of symmetric and asymmetric regimes, transitions into chaotic regimes via an infinite number of period doublings.

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The main difficulty in analytic approaches is the representation of solutions. Currently, we observe that most researchers are using iterative procedures and harmonic balance techniques. We shall discuss the validity of these approaches by exploiting the symmetry properties of Duffing's equation. By applying the Laplace-Borel transformation, we shall show that the first-order approximation of the iterative solution will determine the nature of the solution. Given the first-order approximation, one is still faced with algebraic manipulation of nonlinear terms. Traditional harmonic balancing results in unwieldy calculations when we want to take into account the higher-order approximations. The Laplace-Borel transformation approach improves the harmonic balancing method with a more manageable algebra. We shall show that the Laplace-Borel transformation approach provides an operational calculus similar to that of the Heaviside calculus.

Next, we shall apply this new algebraic technique to elucidate an intricate feature of the nonlinear mechanics of a hard spring, namely, a symmetry-breaking bifurcation. In this case, we shall show that only a second-order harmonic balance approximation is sufficient to capture the essential features of a symmetry-breaking bifurcation. The bifurcation analysis based on Floquet's theory gives values of parameters in reasonable agreement with exact numerical results.

ALGEBRA OF LAPLACE-BOREL TRANSFORMATION

We shall give a brief summary of the principal rules of Laplace-Borel transformation and refer the interested reader to references 1-3 for greater detail. To each analytical function $f(t)$ having an expansion

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \tag{1}$$

there exists a corresponding generating power series G given by:

$$G = \sum_{n=0}^{\infty} a_n x_0^n \tag{2}$$

We have the following algebraic rules which are obtained in references 1-3:

$$\int_0^t d\tau \Longleftrightarrow x_0 \tag{3}$$

The Laplace-Borel transformation operates as if the integral is replaced by the symbolic variable x_0; while in the case of the Laplace transformation, the derivative is replaced by the symbolic variable s. The Laplace-Borel transformation of elementary functions is analogous to the Laplace transformation and we shall use the following relation:

$$e^{jwt} \Longleftrightarrow (1 - jwx_0)^{-1} \tag{4}$$

The Laplace-Borel transformation has the additional capability to handle the product of functions via the so-called shuffle product Ш

$$f(t).g(t) \Longleftrightarrow G(f)\,\text{Ш}\,G(g) \tag{5}$$

where Ш denotes the shuffle product operation. For instance, the shuffle product of two rational simple functions is:

$$(1 - ax_0)^{-1} \text{Ш} (1 - bx_0)^{-1} = (1 - (a + b)x_0)^{-1} \tag{6}$$

The Laplace-Borel transformation treats the forcing term as a symbolic variable x_1

$$\int_0^t f(\tau)\, d\tau \Longleftrightarrow x_1 \tag{7}$$

where f denotes the forcing term. The product of x_1 by x_0 is noncommutative

$$x_1 . x_0 \neq x_0 . x_1 \tag{8}$$

because:

$$\int_0^t d\tau_1 \int_0^{\tau_1} f(\tau_2)\, d\tau_2 \neq \int_0^t f(\tau_1)\, d\tau_1 \int_0^{\tau_1} d\tau_2 \tag{9}$$

At the end of symbolic operations, x_1 is replaced formally by:

$$x_1 . \Rightarrow x_0 . G(f) \text{Ш} \tag{10}$$

SYMMETRY PROPERTY OF DUFFING'S EQUATION

Duffing's equation is given as

$$\ddot{x} + \alpha \dot{x} + w_0^2 x + \beta x^3 = f coswt \tag{11}$$

where $\alpha > 0$ and $\beta \lesseqgtr 0$. Duffing's equation has an invariance property with respect to the following transformation:

$$t \Rightarrow t + \frac{\pi}{w}, \qquad x \Rightarrow -x \tag{12}$$

Due to this property, a symmetric solution x_s is defined as:

$$x_s(t) = -x_s(t + \frac{\pi}{w}) \tag{13}$$

If a symmmetric solution is represented by a Fourier series, the series has only odd terms due to the property of equation (13), i.e.,

$$x_s(t) = \sum_{n=0}^{\infty} A_{2n+1} cos((2n + 1)wt) + \sum_{n=0}^{\infty} B_{2n+1} sin((2n + 1)wt) \tag{14}$$

The symmetric solution has a zero time average. By contrast, the asymmetric solution will contain even terms and hence may have a non-zero time average.

ANALYTICAL REPRESENTATION OF SOLUTIONS

Let us integrate Duffing's equation twice:

$$x(t) + \alpha \int_0^t x(\tau_1)d\tau_1 + w_0^2 \int_0^t d\tau_1 \int_0^{\tau_1} x(\tau_2)d\tau_2 + \beta \int_0^t d\tau_1 \int_0^{\tau_1} x^3(\tau_2)d\tau_2$$
$$= \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 f(\tau_2) + x(0) + (\dot{x}(0) + \alpha x(0)) \int_0^t d\tau_1 \quad (15)$$

where $f(t) = f cos(wt)$ and $x(0), \dot{x}(0)$ are initial displacement and velocity respectively. Next, we shall obtain the Laplace-Borel transformation of this equation by means of the algebraic rules listed earlier:

$$G(x) + \alpha x_0 G(x) + w_0^2 x_0^2 G(x) + \beta x_0^2 G(x) \text{Ш} G(x) \text{Ш} G(x)$$
$$= x_0 x_1 + x_0 + (\dot{x}(0) + \alpha x(0)) x_0 \quad (16)$$

After some algebraic manipulation, we have

$$G(x) = -\beta(1 - a_1 x_0)^{-1} x_0 (1 - a_2 x_0)^{-1} x_0 G(x) \text{Ш} G(x) \text{Ш} G(x)$$
$$+ (1 - a_1 x_0)^{-1} x_0 (1 - a_2 x_0)^{-1} x_1 + (1 - a_1 x_0)^{-1} (1 - a_2 x_0)^{-1} (x(0) + (\dot{x}(0) + \alpha x(0)) x_0) \quad (17)$$

where a_1^{-1} and a_2^{-1} are roots of the equation:

$$1 + \alpha x_0 + w_0^2 x_0^2 = 0 \quad (18)$$

Our interest lies principally in the large-time behavior of the solution. If we neglect the transient terms (i.e., terms that die off with large time) we have to the first order

$$G^{(1)}(x_0) = (1 - a_1 x_0)^{-1} x_0 (1 - a_2 x_0)^{-1} x_0 G(f coswt) \quad (19)$$

or, using the algebraic rule (eq. (4)),

$$G^{(1)}(x_0) = (1 - a_1 x_0)^{-1} x_0 (1 - a_2 x_0)^{-1} x_0 (f/2)(1 - jwx_0)^{-1} + (1 + jwx_0)^{-1}) \quad (20)$$

If we define

$$\Omega_{-1} = \frac{1}{w_0^2 - w^2 + j\alpha w} \quad (21)$$

$$\Omega_{+1} = \frac{1}{w_0^2 - w^2 - j\alpha w} \quad (22)$$

then after decomposition into simple rational fractions and neglecting the transient terms:

$$G^{(1)}(x_0) \simeq (f/2)(\Omega_{-1}(1 - jwx_0)^{-1} + \Omega_{+1}(1 + jwx_0)^{-1}) \quad (23)$$

If we insert $G^{(1)}(x_0)$ into the shuffle product term, we get a second-order approximation. The shuffle product is an operation which can be done on the computer using any one of the symbolic programming languages available to the analyst. In this analysis, however, we shall simply use the algebraic rule (eq.(6)) to obtain:

$$G \text{Ш} G \text{Ш} G = 3\Omega_{-1}^2\Omega_{+1}(1 - jwx_0)^{-1} + 3\Omega_{-1}\Omega_{+1}^2(1 + jwx_0)^{-1}$$
$$+ \Omega_{-1}^3(1 - 3jwx_0)^{-1} + \Omega_{+1}^3(1 + 3jwx_0)^{-1} \tag{24}$$

For the second-order approximation, we see that we get neither constant terms nor even terms in $(\pm njwt)$. This will remain true when iteration is carried out to higher order. We conclude that a solution based on iteration of the linear solution will be capable of representing only the first symmetric regime.

Next, we shall discuss the validity of the harmonic balance method. The harmonic balance method is based on the truncated Fourier series:

$$x(t) = \sum_{n=0}^{N} A_n cos(nwt) + \sum_{n=0}^{N} B_n sin(nwt) \tag{25}$$

For the first-order approximation of a symmetric solution, we have:

$$x_s(t) = A_1 cos(wt) + B_1 sin(wt) \tag{26}$$

When we substitute equation (26) into equation (11) and compute the coefficients A_1, B_1 by matching the terms multiplied by $cos(wt)$ and $sin(wt)$ and neglecting the higher-order terms multiplied by $cos(nwt)$ and $sin(nwt)$ with $n > 1$, we obtain the solution via harmonic balance. Computation of the magnitudes of the neglected terms involves some algebraic manipulation that becomes unwieldy for higher-order approximations. The Laplace-Borel transformation has the merit of involving relatively easy algebraic manipulation that bears a strong analogy to the Heaviside operational calculus. Let us illustrate for the first-order approximation

$$x_s \Longleftrightarrow G(x_s) \tag{27}$$

$$G(x_s) = A_{-1}(1 - jwx_0)^{-1} + A_{+1}(1 + jwx_0)^{-1} \tag{28}$$

where:

$$A_{\pm 1} = (A_1 \pm jB_1)/2 \tag{29}$$

If furthermore we neglect the transient terms, we have:

$$G(x_s) = -\beta(1 - a_1x_0)^{-1}x_0(1 - a_2x_0)^{-1}G(x_s)\text{Ш}G(x_s)\text{Ш}G(xs)$$
$$+ (1 - a_1x_0)^{-1}x_0(1 - a_2x_0)^{-1}x_0(f/2)((1 - jwx_0)^{-1} + (1 + jwx_0)^{-1}) \tag{30}$$

Decomposition into simple rational fractions yields:

$$A_{-1}(1-jwx_0)^{-1}+A_{+1}(1+jwx_0)^{-1}$$
$$=\frac{f/2}{(w_0^2-w^2)+j\alpha w}((1-jwx_0)^{-1}+(1+jwx_0)^{-1})-\beta(\frac{3A_{-1}^2A_{+1}(1-jwx_0)^{-1}}{(w_0^2-w^2)+j\alpha w}$$
$$+\frac{3A_{-1}A_{+1}^2(1+jwx_0)^{-1}}{(w_0^2-w^2)+j\alpha w}+\frac{A_{-1}^3(1-j3wx_0)^{-1}}{(w_0^2-9w^2)+j\alpha 3w}+\frac{A_{+1}^3(1+3jwx_0)^{-1}}{(w_0^2-9w^2)-j\alpha 3w}) \tag{31}$$

Here the important point is that we can identify the amplitude ratio of the component of [1w] to that of [3w] through the factor β

$$\frac{9}{(w_0^2-w^2)^2+(\alpha w)^2}\div\frac{1}{(w_0^2-9w^2)^2+(\alpha 3w)^2} \tag{32}$$

with algebraic manipulation similar to that of the Heaviside operational calculus. Hence, we are able to find the parameter range in which the component in [3w] can be neglected.

SYMMETRY-BREAKING BIFURCATION OF A HARD SPRING

The idea underlying the study of symmetry-breaking bifurcation originates from the problem of predicting critical values of parameters at onset of chaos. The Melnikov criterion predicts onset of chaos in the case of a soft spring with some limited success (ref. 8); however it cannot be applied to the case of a hard spring. A natural alternative approach is then to predict critical values of parameters for the first few bifurcations, on the hypothesis that this may be sufficient to establish the trend toward the onset of chaos.

It has been found (ref. 4) that the first-order approximation of the harmonic balance method cannot allow a symmetry-breaking bifurcation, while the second-order approximation may do so. Therefore, the validity of representing a solution by a Fourier series that is truncated at some order becomes questionable (ref. 4). There is no general theory that would allow us to affirm the validity of a truncated series solution. However, we shall show that a second-order harmonic balance representation leads to values of critical parameters at a symmetry-breaking bifurcation that are in reasonable agreement with numerical results based on a complete integration of Duffing's equation.

In order to visualize the symmetry-breaking bifurcation, we define the amplitude as $< x^2 >$- $< x >^2$, where $<>$ denotes the average over a period. We draw the amplitude response curve versus frequency of the symmetric solution of Duffing's equation represented by

(i) a first-order approximation

$$x_s^{(1)}=A_1coswt+B_1sinwt \tag{33}$$

(ii) the response curve of the symmetric solution represented by a second-order approximation

$$x_s^{(2)}=A_1'coswt+B_1'sinwt+A_3'cos3wt+B_3'sin3wt \tag{34}$$

and (iii) the response curve of the asymmetric solution represented by:

$$x_{as} = A_0'' + A_1'' cos wt + B_1'' sin wt + A_2'' cos 2wt + B_2'' sin 2wt + A_3'' cos 3wt + B_3'' sin 3wt \quad (35)$$

The principal steps in using the Laplace-Borel transformation are as follows.

1. Calculate the shuffle product.

2. Decompose into simple rational fractions.

3. Match coefficients of terms $(1 - jnwx_0)^{-1}$, where n is an integer.

4. Solve numerically for the coefficients A_n, B_n.

The results are given in figure 1 for the following values of parameters: $\alpha = 0.2, w_0 = 1.0$, $f = 10.0$.

The response curve of the symmetric solution represented by equation (32) consists of only one main resonance peak, while the one represented by equation (33) consists of a main resonance peak R_1 and a secondary resonance peak R_3. The response curve of the asymmetric solution represented by equation (34) consists of only a resonance peak R_2, at a frequency range between R_1 and R_3. In contrast to the case of a soft spring where the asymmetric solution is present for a wide range of frequency w and even for small values of the forcing amplitude f, the asymmetric solution for the hard spring is confined to a small range of w. Furthermore, it does not appear for small values of f. The difference can be viewed as originating from the difference in form of the potential well for the hard spring and for the soft spring. In the former case, the potential consists of a single symmetric well with infinite amplitude at infinite distance, while in the latter case, the potential well is confined to finite values. For the infinite symmetric well, the frequency has to be tuned to some frequency in order to break the symmetry of the motion, even at sufficient values of f. A complete numerical computation would include more resonance peaks; however, the main features at frequency $w > 0.6$ of figure 1 are in qualitative agreement with numerical results of reference 6. Using the second-order harmonic balance approximation, we study its symmetry-breaking bifurcation by stability analysis based on Floquet's theory. Our approach incorporates the mathematical works on Hill's equation (ref. 7). The results are given for comparison with numerical results from reference 4 in figure 2 (Values of parameters: $\alpha = 0.2, \beta = (2)^{3/2}/3$).

We note first that we are able to reproduce one band of values of parameter f (usual approaches [4,5] reproduce only a lower limit value for f). Our limitation to one band of values originates from the second-order approximation for the symmetric solution. The values of f obtained for low values of w ($w < 0.9$) are incorrect. This is expected because an exact solution would include more harmonics and the amplitude of oscillation at low w is then affected. The results obtained for f at high values of w are in reasonable agreement with numerical results.

CONCLUSIONS

We have applied the Laplace-Borel transformation to discuss representation of analytical solutions of Duffing's equation. Exploiting the symmetry property of this equation, we have demonstrated the following.

1. In an iterative procedure, use of the first order iteration based on the linear equation leads only to a description of the first regime (symmetric solution).

2. Implementing the harmonic balance method with the Laplace-Borel transformation gives an algebra very similar to that of the Heaviside operational calculus.

Analysis of the nonlinear mechanics of a hard spring by means of the Laplace-Borel transformation gives values of parameters at a symmetry-breaking bifurcation in reasonable agreement with numerical results. It should be possible to implement the Laplace-Borel transformation on computers as a combination of symbolic and numerical computation in order to predict values of parameters for the first few period-doubling bifurcations.

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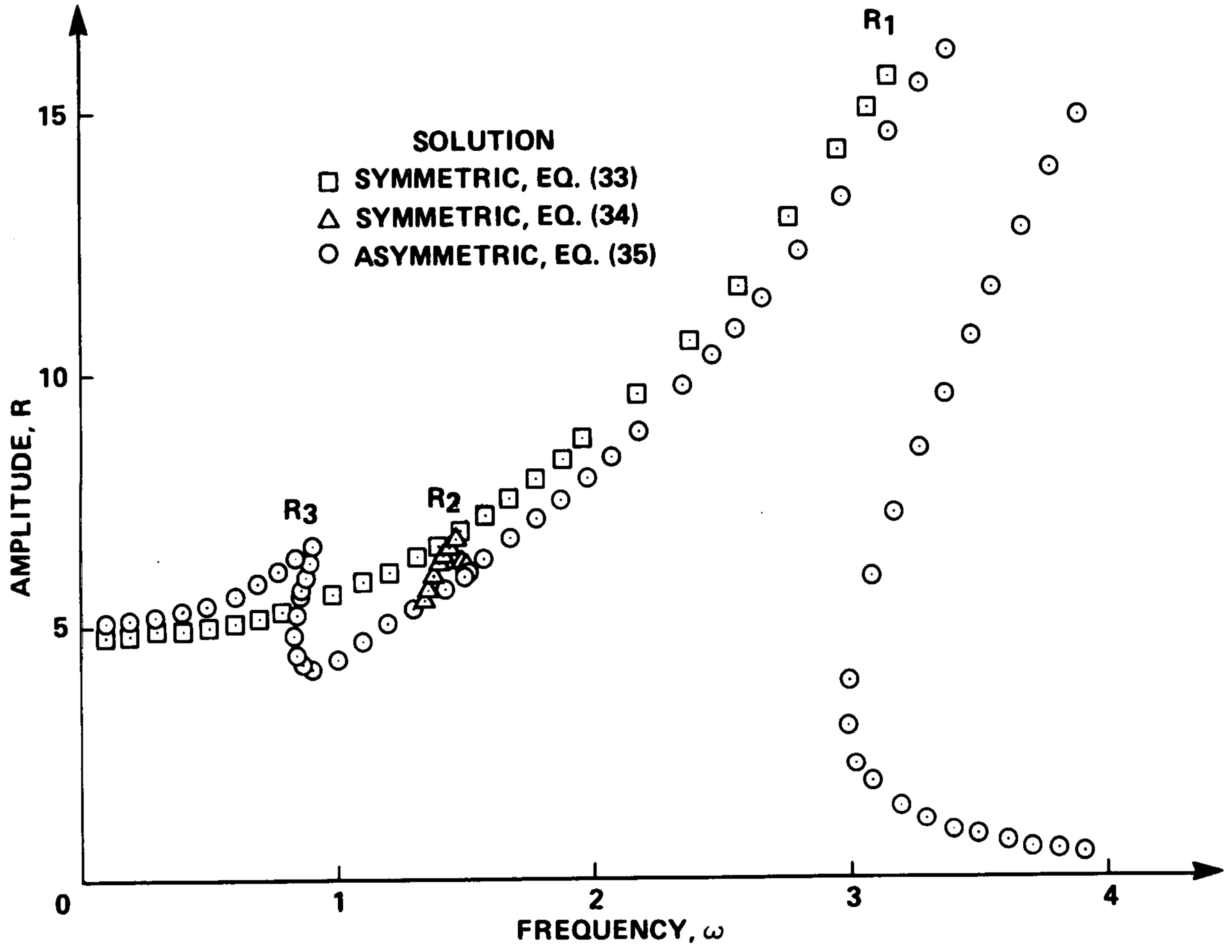

Figure 1.- Amplitude response curve versus frequency for Duffing's equation: $\ddot{x} + 0.2\dot{x} + x + x^3 = f cos(wt)$.

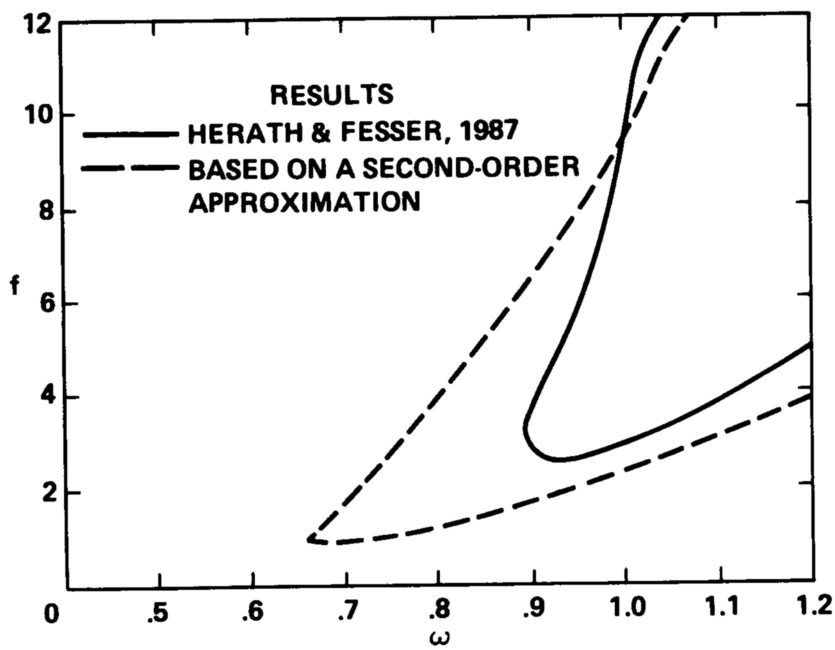

Figure 2.- Values of forcing amplitude f versus frequency w for bifurcation to symmetry-breaking.

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16. Abstract

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